3. V. Ya. Shkadov, "Single waves in a layer of viscous liquid," Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza, No. 1 (1977).
4. S. V. Alekseenko, V. E. Nakoryakov, and B. G. Pokusaev, "Waves at the surface of a vertically down-flowing film of liquid," Preprint Inst. Tekh. Fiz., Sib. Otd., Akad. Nauk SSSR, No. 36-79, Novosibirsk (1979).

UNSTEADY ROTATION OF A CYLINDER IN A VISCOUS FLUID

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UDC 532.516

The flow of a viscous fluid around a cylinder set in rotational motion at constant angular velocity was investigated in [1, 2]. The present paper deals with the problem of rotation, in a viscous incompressible fluid, of a round cylinder, on unit length of which, beginning at time t = 0, there acts a constant moment of external forces M. The fluid flow is assumed to be plane. At $t \leqslant 0$ the cylinder and fluid are at rest.

We select cylindrical coordinates r, θ, and z in such a way that the z axis is directed along the cylinder axis. We assume that the flow velocity V is independent of θ. Then, as is easily verified, the r component of the vector V is zero, and the considered problem reduces to solution of the equations

$$\frac{\partial V_\theta}{\partial t} = \nu\left(\frac{\partial^2 V_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial V_\theta}{\partial r} - \frac{V_\theta}{r^2}\right); \tag{1}$$

$$I\,d\Omega/dt = M + L \tag{2}$$

with the following initial and boundary conditions:

$$V_\theta = 0 \text{ when } t = 0,\ r \geqslant a; \tag{3}$$

$$V_\theta = a\Omega \text{ when } r = a; \tag{4}$$

$$V_\theta \to 0 \text{ when } r \to \infty, \tag{5}$$

where V_θ is the θ component of vector V; I is the moment of inertia of unit length of the cylinder; Ω is the angular velocity of the cylinder; a is the radius of the cylinder; ν is the kinematic viscosity; $L = 2\pi\mu a^2(\partial V_\theta/\partial r|_{r=a} - \Omega)$ is the moment of viscous forces acting on unit length of the cylinder due to the fluid; $\mu = \rho\nu$; ρ is the density of the fluid.

To solve the posed problem we use an operational method. Converting to images in (1), (2), (4), and (5), we obtain

$$\frac{\partial^2 V_\theta^*}{\partial r^2} + \frac{1}{r}\frac{\partial V_\theta^*}{\partial r} - \left(\frac{1}{r^2} + \frac{p}{\nu}\right)V_\theta^* = 0; \tag{6}$$

$$Ip\Omega^* = M^* + L^*; \tag{7}$$

$$V_\theta^* = a\Omega^* \quad \text{when } r = a; \tag{8}$$

$$V_\theta^* \to 0 \quad \text{when } r \to \infty, \tag{9}$$

where

$$V_\theta^* = \int_0^\infty e^{-pt} V_\theta dt; \quad \Omega^* = \int_0^\infty e^{-pt}\Omega dt;$$

$$M^* = \frac{M}{p}, \quad L^* = 2\pi\mu a^2\left(\left.\frac{\partial V_\theta^*}{\partial r}\right|_{r=a} - \Omega^*\right); \tag{10}$$

p is a complex variable.

The solution of Eq. (6) satisfying conditions (8), (9) has the form

$$V_\theta^* = a\Omega^* \frac{K_1\left(r\frac{p^{1/2}}{\nu^{1/2}}\right)}{K_1\left(a\frac{p^{1/2}}{\nu^{1/2}}\right)}, \tag{11}$$

where K_1 is a MacDonald function. Using (7), (10), (11), we obtain

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 3, pp. 66-69, May-June, 1980. Original article submitted June 12, 1979.

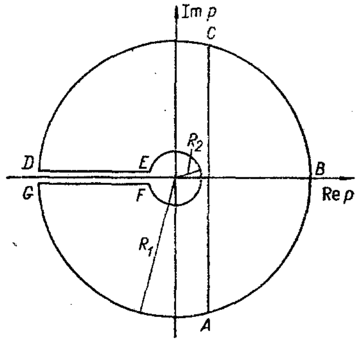

Fig. 1

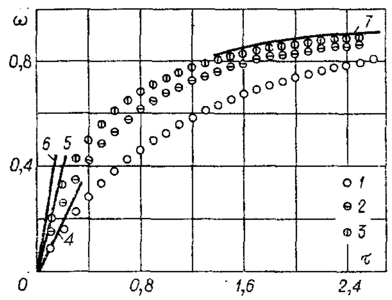

Fig. 2

$$\Omega^* = \frac{M}{p\Phi} K_1\left(a\frac{p^{1/2}}{\nu^{1/2}}\right),$$

where

$$\Phi = 2\pi\mu a^2 \left\{\left(1 + \frac{Ip}{2\pi\mu a^2}\right) K_1\left(a\frac{p^{1/2}}{\nu^{1/2}}\right) - a\,\frac{\partial K_1\left(r\frac{p^{1/2}}{\nu^{1/2}}\right)}{\partial r}\Bigg|_{r=a}\right\}.$$

Thus, for V_θ we have the following expression:

$$V_\theta = \frac{aM}{2\pi i}\int\limits_{\alpha-i\infty}^{\alpha+i\infty} e^{pt}\,\frac{K_1\left(r\frac{p^{1/2}}{\nu^{1/2}}\right)}{p\Phi}\,dp,$$

where the integral is taken over the straight line Re p = α; $\alpha > 0$.

The function Φ is nonzero over the entire complex plane of the variable p with a branch cut along the negative part of the real axis [3]. According to the Cauchy theorem

$$\int\limits_{ABCA} \frac{K_1\left(r\frac{p^{1/2}}{\nu^{1/2}}\right)}{p\Phi}\,dp = 0; \tag{12}$$

$$\int\limits_{ACDEFGA} e^{pt}\,\frac{K_1\left(r\frac{p^{1/2}}{\nu^{1/2}}\right)}{p\Phi}\,dp = 0, \tag{13}$$

where the integration is taken over the contours illustrated in Fig. 1. Converting in (12) to the limit $R_1 \to \infty$, and in (13) to $R_1 \to \infty$, $R_2 \to 0$, we obtain

$$\int\limits_{\alpha-i\infty}^{\alpha+i\infty} \frac{K_1\left(r\frac{p^{1/2}}{\nu^{1/2}}\right)}{p\Phi}\,dp = 0; \tag{14}$$

$$V_\theta = \frac{M}{4\pi\mu r}\left\{1 + \frac{4\varkappa s}{\pi}\int\limits_0^\infty \frac{e^{-\tau\xi^2}P}{\xi^2 Q}\,d\xi\right\}, \tag{15}$$

where

$$P = N_1(s\xi)\left[\xi J_1(\xi) - \varkappa J_2(\xi)\right] - J_1(s\xi)\left[\xi N_1(\xi) - \varkappa N_2(\xi)\right];$$
$$Q = \left[\xi J_1(\xi) - \varkappa J_2(\xi)\right]^2 + \left[\xi N_1(\xi) - \varkappa N_2(\xi)\right]^2;$$

$\varkappa = 2\pi\rho a^4/I$; $s = r/a$; $\tau = \nu t/a^2$; J_1, J_2, N_1, N_2 are Bessel and Neumann functions. When t = 0 expression (15) for V_θ becomes zero [this follows from (14)] in accordance with condition (3).

The angular velocity of the cylinder is

$$\Omega = \frac{M}{4\pi\mu a^2}\left\{1 - \frac{8\varkappa^2}{\pi^2}\int\limits_0^\infty \frac{e^{-\tau\xi^2}}{\xi^3 Q}\,d\xi\right\}. \tag{16}$$

Using the known expansions of the Bessel and Neumann functions in series [4], we can obtain from (16) the following asymptotic relations:

$$\omega \sim 1 - 1/4\tau \text{ for } \tau \to \infty, \ \omega \sim f(\varkappa)\tau \text{ for } \tau \to 0,$$

where

$$\omega = \frac{4\pi\mu a^2\Omega}{M}; \quad f(\varkappa) = \frac{8\varkappa^2}{\pi^2}\int\limits_0^\infty \frac{d\xi}{\xi Q}.$$

According to the Cauchy theorem

$$\int\limits_{ABCDEFGA} \frac{K_1\left(a\frac{p^{1/2}}{\nu^{1/2}}\right)}{\Phi}\, dp = 0. \tag{17}$$

Converting in (17) to the limit $R_1 \to \infty$, $R_2 \to 0$, we obtain

$$\int\limits_0^\infty \frac{d\xi}{\xi Q} = \frac{\pi^2}{4\varkappa}.$$

Thus,

$$\omega \sim 2\varkappa\tau \text{ when } \tau \to 0.$$

Figure 2 shows ω as a function of τ for $\varkappa$ = 0.5, 1, 1.5 (points 1-3). The numbers 4-7 denote the lines $\omega = \tau$, $\omega = 2\tau$, $\omega = 3\tau$, $\omega = 1 - 1/4\tau$.

I express my thanks to L. Ya. Rybak for performing the numerical calculations from Eq. (16).

LITERATURE CITED

1. N. A. Slezkin, Dynamics of a Viscous Incompressible Fluid [in Russian], GITTL, Moscow (1955).
2. D. D. Mallick, "Nonuniform rotation of an infinite circular cylinder in an infinite viscous liquid," Z. Angew. Math. Mech., ZAMM, 37, No. 9/10 (1957).
3. A. Erdélyi and W. O. Kermack, "Note on the equation $f(z)K_n'(z) - g(z)K_n(z) = 0$," Proc. Camb. Phil. Soc., 41, 74 (1945).
4. B. G. Korenev, Introduction to the Theory of Bessel Functions [in Russian], Nauka, Moscow (1971).